THE AXISYMMETRIC PROBLEM OF PROPAGATION OF A TENSION CRACK

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In this paper the three-dimensional (axisymmetric) problem of nonsteady propagation of a crack in an elastic medium under the influence of a homogeneous tensile stress is solved. The analogous plane problem was considered in [1]. An analysis of the formulation of the problem of [1] and the results is given in [2] based on consideration of the cohesive forces acting near the edge of the crack. In this work an equation was obtained for the speed of propagation of the crack. A comparison with the experimental results of Wells and Post [3] may be found in the same reference. The plane problem of propagation of a crack after a semi-infinite cut is instantaneously made in a stressed medium is solved in [4].

The solution of the axisymmetric problem is carried out below. Formulas are obtained for the displacement at the surface of the crack and for stresses near the edge. It is shown that, just as in the plane problem, the speed of propagation of a crack cannot exceed the Rayleigh surface wave velocity. An equation which determines the speed of propagation of a crack is obtained.

1. Formulation of the problem. Let an unbounded elastic medium having shear modulus μ and longitudinal and transverse wave velocities a and b, respectively, be in a state of homogeneous tension for $t < 0$, so that only a single component, σ_z°, of the stress tensor is nonzero. A crack is formed at the instant $t = 0$ at the origin of coordinates. The crack then propagates in the plane $z = 0$ with constant velocity α. For $t > 0$, the surface of the crack in a cylindrical system of coordinates r, φ, z is defined by the relations (Fig.1)

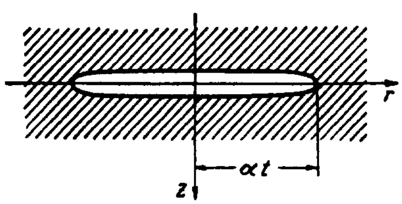

Fig. 1

$$z = 0, \qquad r \leqslant \alpha t \tag{1.1}$$

The surface of the crack is to be free from stresses (we shall neglect the forces of molecular conesion acting near the edge of the crack, considering this region as infinitesimally narrow). Therefore, the elastic disturbances caused by the propagation of the crack must satisfy the conditions

$$\sigma_z = -\sigma_z^{\circ}, \quad \tau_{rz} = 0, \quad \tau_{\varphi z} = 0 \quad \text{for } z = 0, \quad 0 \leqslant r < \alpha t \tag{1.2}$$

on the surface of the crack.

These disturbances are absent at the initial instant of time, which is expressed by the homogeneity of the initial conditions

$$\mathbf{u} = \mathbf{u}^{\cdot} = 0 \quad \text{for } t = 0 \tag{1.3}$$

where $\mathbf{u}$ is the displacement vector with components u_r, u_φ and u_z. It is obvious here that $u_\varphi \equiv 0$, and that all the remaining quantities are independent of φ (axisymmetric problem). The dot denotes differentiation with resect to time.

In addition to the boundary and initial conditions, it is necessary to impose a further condition on the behavior of the solution in the neighborhood of the edge of the crack. This additional condition which restricts the order of growth of stresses near the edge of the crack can be obtained by considering the additional stresses generated by the forces of cohesion [2]. A somewhat different method is proposed below. It is clear physically that a certain amount of energy is dissipated upon formation of the crack. Using the enegy integral of the equations of motion applicable to the total field, it can be established that the rate of energy dissipation (the power expended in formation of the crack) is equal to

$$W = \lim_{\delta \to 0} \iint_{S_\delta} \left\{ \mathbf{t}_n \mathbf{v} - \frac{1}{2} \left[\rho \left(\mathbf{v} \right)^2 + \tau \varepsilon \right] \alpha \cos \left(n, r \right) \right\} dS \tag{1.4}$$

where ε and τ are the strain and stress tensors, $\mathbf{v} \equiv \mathbf{u}^{\cdot}$ is the velocity vector of the particles of the medium, the surface S_δ surrounds the edge of the crack and is at a distance δ from it, and n is the outer normal to S_δ. Thus, we require that the integral in (1.4) approach a finite positive (nonzero) limit independently of the way S_δ shrinks down to the edge of the crack. The second term in the braces, which is related to the motion of the surface of integration, gives zero in the limit. This may be shown by choosing a surface S_δ, whose intersection with the rz plane has the form of a rectangle with sides $2\delta_1$ and $2\delta_2$ in the r and z directiors, respectively, and by letting δ_2 approach zero for fixed δ_1. Thus, the required condition has the form

$$0 < 2\pi \alpha t \lim_{\delta \to 0} \int_{l_\delta} \mathbf{t}_n \mathbf{v} \, dl < \infty \tag{1.5}$$

where l_δ is the section of S_δ by the rz plane. Here the symmetry of the problem with respect to the z-axis has been used. It will be shown in Section 3 that the stress and velocity components have singularities of the same order for $r \to \alpha t$ and $z \to 0$. In order that the integral (1.5) approach a finite limit it is, therefore, necessary that the stress and velocity components (or at least some of them) increase as $\delta^{-\frac{1}{2}}$ near the edge of the crack.

It is easy to see that the stress and velocity components must be homogeneous functions of the coordinates and time of order zero. From this and the requirement on the rates of increase which has been shown, it follows that the asymptotic expressions for the components of velocity and stress must be proportional to $\sqrt{t/\delta}$ as $\delta \to 0$. The integral in (1.5) then turns out to be proportional to t, i.e. $W \sim t^2$. This conclusion seems strange since the surface of the crack increases at the rate $4\pi\alpha^2 t$ so that the energy is not dissipated proportionally to the area. This may be explained in the following way (see also [2]). It can be assumed that the edge of the crack is surrounded by a region in which plastic deformation of the material takes place. In the present problem this region is considered to be infinitesimally small; however, it actually has small but finite dimensions. Since the plastic region is absent at the initial instant, its dimensions must increase at the same time as the crack grows, until they attain some stationary magnitude. It may be considered that for small values of t the dimensions of the plastic region increase at a constant rate (proportional to the rate of propagation of the crack, α) and that the energy expended in forming the plastic region increases proportionally to its volume. We may then set

$$W = 2\pi\, \alpha^3 t^2 C \quad (C = \text{const})$$

We obtain the additional condition in the form

$$\lim_{\delta\to 0} \int_{l_\delta} \mathbf{t}_n \mathbf{v}\, dl = \alpha^2 t C \tag{1.6}$$

instead of (1.5).

We note that the variation of stresses in the neighborhood of the edge of the crack proportional to $\sqrt{t/\delta}$ was also obtained in [1 and 4], and has been confirmed experimentally as well [2 and 3]. It is clear from the above explanation that the assumption of a constant velocity of propagation of the crack is valid only for the initial stage of crack growth. This conclusion was arrived at [2] from somewhat different considerations.

It is convenient to reduce the problem to a boundary value problem for the half-space $z > 0$. To do this we note that on passing through the plane $z = 0$ the stresses must be continuous everywhere and the displacements continuous outside the crack. We split the displacement vector into symmetric and antisymmetric parts relative to the plane $z = 0$. In the antisymmetric part, u_z and τ_{rz} are even and u_r and σ_z are odd functions of z . Using the indicated requirement of continuity and conditions (1.2) we see that u_r and σ_z in the antisymmetric part should vanish in the entire plane $z = 0$. This, together with the initial conditions, shows that the antisymmetric part is identically equal to zero, i.e. that the solution of the problem is symmetric with respect to the plane $z = 0$. Then u_z and τ_{rz} are odd and u_r and σ_z are even with respect to z . This gives the boundary conditions for $z = 0$

$$\begin{aligned} &\tau_{rz} = 0 && \text{for } z = 0, && 0 \leqslant r < \infty \\ &\sigma_z = -\sigma_z^\circ && \text{for } z = 0, && 0 \leqslant r < \alpha t \\ &u_z = 0 && \text{for } z = 0, && \alpha t < r < \infty \end{aligned} \tag{1.7}$$

2. Solution of the problem. The solution of a selfsimilar axisymmetric problem, when the components of velocity and stress are homogeneous functions of the coordinates and time of order zero, under the condition

$$\tau_{rz} = 0 \quad \text{for } z = 0, \quad 0 \leqslant r < \infty \tag{2.1}$$

can be written in the form [5]

$$\begin{aligned} u_r^\cdot \equiv v_r = v_r^{(1)} + v_r^{(2)}, &\qquad v_r^{(1,2)} = \mathrm{Re} \int_{-\pi}^{\pi} V_\xi^{(1,2)} (\vartheta^{(1,2)}) \cos \Omega \, d\Omega \\ u_z^\cdot \equiv v_z = v_z^{(1)} + v_z^{(2)}, &\qquad v_z^{(1,2)} = \mathrm{Re} \int_{-\pi}^{\pi} V_z^{(1,2)} (\vartheta^{(1,2)}) \, d\Omega \\ \tau_{rz} = \tau_{rz}^{(1)} + \tau_{rz}^{(2)}, &\qquad \tau_{rz}^{(1,2)} = \mathrm{Re} \int_{-\pi}^{\pi} T_{\xi z}^{(1,2)} (\vartheta^{(1,2)}) \cos \Omega d\, \Omega \end{aligned} \tag{2.2}$$

$$\sigma_z = \sigma_z^{(1)} + \sigma_z^{(2)}, \qquad \sigma_z^{(1,2)} = \mathrm{Re}\int_{-\pi}^{\pi} \Sigma_z^{(1,2)}(\vartheta^{(1,2)})\, d\Omega \tag{2.2 cont.}$$

where $\vartheta^{(1)}$ and $\vartheta^{(2)}$ are determined from Equations

$$\begin{aligned} \delta^{(1)} &\equiv t - \vartheta^{(1)} r\cos\Omega - z\sqrt{a^{-2} - \vartheta^{(1)2}} = 0 \\ \delta^{(2)} &\equiv t - \vartheta^{(2)} r\cos\Omega - z\sqrt{b^{-2} - \vartheta^{(2)2}} = 0 \end{aligned} \tag{2.3}$$

and the functions under the integral signs are expressed in terms of a single unknown function $F(\vartheta^2)$ by the relations

$$\begin{aligned} &V_\xi^{(1)\prime}(\vartheta) = \frac{2\vartheta^2(1-2b^2\vartheta^2)}{\sqrt{a^{-2}-\vartheta^2}} F'(\vartheta^2), \quad V_\xi^{(2)\prime}(\vartheta) = -4b^2\vartheta^2\sqrt{b^{-2}-\vartheta^2}\, F'(\vartheta^2) \\ &V_z^{(1)\prime}(\vartheta) = 2\vartheta(1-2b^2\vartheta^2) F'(\vartheta^2), \quad V_z^{(2)\prime}(\vartheta) = 4b^2\vartheta^3 F'(\vartheta^2) \\ &\frac{1}{2\mu}\Sigma_z^{(1)\prime}(\vartheta) = -\frac{4b^2\vartheta(\vartheta^2 - {}^1/_2 b^2)^2}{\sqrt{a^{-2}-\vartheta^2}} F'(\vartheta^2) \\ &\frac{1}{2\mu}\Sigma_z^{(2)\prime}(\vartheta) = -4b^2\vartheta^3\sqrt{b^{-2}-\vartheta^2}\, F'(\vartheta^2) \\ &\frac{1}{2\mu} T_{\xi z}^{(1)\prime}(\vartheta) = -\frac{1}{2\mu} T_{\xi z}^{(2)\prime}(\vartheta) = -2\vartheta^2(1-2b^2\vartheta^2) F'(\vartheta^2) \end{aligned} \tag{2.4}$$

Primes denote differentiation of the functions with respect to their arguments (i.e., for instance, $V_\xi^{(1)\prime}(\vartheta) = dV_\xi^{(1)}/d\vartheta$, but $F'(\vartheta^2) = dF/d(\vartheta^2)$). This solution is obtained by the method of V.I. Smirnov and S.L. Sobolev (*).

For $z = 0$ the functions $\vartheta^{(1)}$ and $\vartheta^{(2)}$ assume the same values,

$$\vartheta^{(1)} = \vartheta^{(2)} = t/r\cos\Omega = \vartheta.$$

Differentiating the second and fourth expressions of (2.3) with respect to time, setting $z = 0$, using (2.4), and transforming to the new variable $\nu = \vartheta^2$, we obtain

$$\frac{r}{2} v_z^{\cdot} = \mathrm{Re}\int_{l_\nu} F'(\nu)\frac{d\nu}{\sqrt{\nu-\nu_0}} \quad \text{for } z=0 \tag{2.5}$$

$$\frac{r}{2}\sigma_z^{\cdot} = -4\mu b^2\,\mathrm{Re}\int_{l_\nu} \frac{R(\nu)}{\sqrt{a^{-2}-\nu}} F'(\nu)\frac{d\nu}{\sqrt{\nu-\nu_0}}$$

$$\nu_0 = t^2/r^2, \qquad R(\nu) = (\nu - {}^1/_2 b^{-2}) + \nu\sqrt{a^{-2}-\nu}\sqrt{b^{-2}-\nu} \tag{2.6}$$

with the path of integration shown in Fig.2. We define the principal values of the radicals occuring in (2.5) by making suitable cuts along the positive real axis and by requiring that $\sqrt{a^{-2}-\nu}$ and $\sqrt{b^{-2}-\nu}$ are positive, and $\sqrt{\nu-\nu_0}$ equals $i\sqrt{\nu_0}$ for $\nu = 0$.

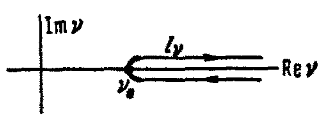

Fig. 2

*) Chapt. XII of [6].

In order to satisfy the initial conditions we require that (2.5) vanish for $r > at$, i.e. for $\nu_0 < 1/a^2$. For this it is necessary that the integrands decrease sufficiently rapidly at infinity and that it is possible to deform the path of integration freely in the half plane $\operatorname{Re}\nu < \nu_0$ for $\nu_0 < 1/a^2$. It follows from this that $F'(\nu)$ must be regular except at the cut from $1/a^2$ to ∞ and must decrease faster than ν^{-1} as $\nu \to \infty$. Further, it follows from the boundary conditions (1.7) that the first of the expressions of (2.5) vanishes for $r > \alpha t$, i.e. for $\nu_0 < 1/\alpha^2$. For this to occur, the function $F'(\nu)$ should be regular for $\operatorname{Re}\nu < 1/\alpha^2$. The second expression of (2.5) must disappear for $r < \alpha t$, i.e. for $\nu_0 > 1/\alpha^2$ for this to occur the integrand must be regular and singlevalued for $\operatorname{Re}\nu > \nu_0 > 1/\alpha^2$. These considerations make it possible to find $F'(\nu)$. The final expression for $F'(\nu)$ depends on the magnitude of α. Let us first examine the case $0 < \alpha < c$, where c is the Rayleigh surface wave velocity $(R(1/c^2) = 0)$. In this case the conditions which have been enumerated are satisfied by Expression

$$F'(\nu) = \frac{A(\nu)}{(\alpha^{-2} - \nu)^n}$$

where n is an integer and $A(\nu)$ is an integral function which does not vanish for $\nu = 1/\alpha^2$. To determine n it is necessary to return to the additional condition (1.6) which, in particular, requires that the function σ_z increase like $\delta^{-\frac{1}{2}}$ as $\delta = |r - \alpha t| \to 0$. Therefore $\sigma_z^{\cdot}$ must behave like $\delta^{-3/2}$, and this will occur only for $n = 2$. Taking into account that $F'(\nu) = o(\nu^{-1})$ for $\nu \to \infty$, we conclude that $A(\nu)$ must be bounded and that $A(\nu) = A = \text{const}$.

Thus

$$F'(\nu) = \frac{A}{(\alpha^{-2} - \nu)^2} \tag{2.7}$$

The following expressions can be obtained analogously to (2.5):

$$\sigma_z = -4\mu b^2 \sqrt{\nu_0} \operatorname{Re} \int_{l_\nu} G(\nu) \frac{d\nu}{\nu \sqrt{\nu - \nu_0}}, \qquad v_z = \sqrt{\nu_0} \operatorname{Re} \int_{l_\nu} F(\nu) \frac{d\nu}{\nu \sqrt{\nu - \nu_0}} \qquad \text{for } z = 0 \tag{2.8}$$

where

$$F(\nu) = \int_0^\nu F'(\lambda)\, d\lambda, \qquad G(\nu) = \int_0^\nu \frac{R(\lambda)}{\sqrt{a^{-2} - \lambda}} F'(\lambda)\, d\lambda \tag{2.9}$$

Here the lower limit of integration is chosen equal to zero so that the point $\nu = 0$ is not a pole of the integrand in (2.8), which is necessary in order to satisfy the boundary conditions. The function $F(\nu)$ may be computed directly and is equal to

$$F(\nu) = \frac{\alpha^2 \nu A}{\alpha^{-2} - \nu} \tag{2.10}$$

However, $G(\nu)$ is transformed in the following way

$$G(\nu) = \int_0^{-\infty} \frac{R(\nu)}{\sqrt{a^{-2}-\nu}} F'(\nu)\, d\nu + \int_{\infty}^{\nu} \frac{R(\lambda)}{\sqrt{a^{-2}-\lambda}} F'(\lambda)\, d\lambda = M + G_1(\nu) \quad (2.11)$$

It is now clear from (2.8) that v_z is actually equal to zero for $r = \alpha t$. Considering $G(\nu)$ in the form (2.11) we see that $G_1(\nu)$ changes sign upon passing through the cut from $1/\alpha^2$ to ∞, and that the integral of this term in (2.8) vanishes for $\nu_0 > 1/\alpha^2$. Therefore,

$$\sigma_z = -4\mu b^2 \int_{l_\nu} \frac{d\nu}{\nu\sqrt{\nu-\nu_0}} M\sqrt{\nu_0} = -8\mu\pi b^2 M \qquad \text{for } z=0,\ r<\alpha t$$

This value should be equal to $-\sigma_z^\circ$, i.e.

$$-\sigma_z{}^\circ = 8\mu\pi b^2 A \int_0^{\infty} \frac{(\nu + {}^1/_2 b^{-2}) - \nu\sqrt{a^{-2}+\nu}\sqrt{b^{-2}+\nu}}{(\alpha^{-2}+\nu)^2\sqrt{a^2+\nu}}\, d\nu \qquad (2.12)$$

From this relation we obtain the value of A. Since A is negative, we set

$$A_1 = -2\pi A \qquad (2.13)$$

We now find the expression for v_z at $z = 0$

$$v_z = \frac{\alpha^3 A_1 t}{\sqrt{\alpha^2 t^2 - r^2}} \qquad \text{for } z=0,\ r<\alpha t \qquad (2.14)$$

Integrating this expression with respect to time, we obtain an expression for the displacement of the surface of the crack

$$u_z = \alpha A_1 \sqrt{\alpha^2 t^2 - r^2} \qquad (2.15)$$

The condition (1.6) still remains to be satisfied. This will be done in the following section.

Let us now turn to the case $c < \alpha < b$. The zero of the function $R(\nu)$, i.e. the point $\nu = 1/c^2$ now lies to the right of the point $\nu = 1/\alpha^2$. Therefore $F'(\nu)'$ can be taken in the form

$$F'(\nu) = \frac{A(\nu)}{(c^2-\nu)(\alpha^{-2}-\nu)^n} \qquad (2.16)$$

As in the preceding case we conclude that $n = 2$, but now $A(\nu)$ turns out to be a linear function, which we write in the form

$$A(\nu) = A(c^{-2}-\nu) + B(\alpha^{-2}-\nu)$$

so that

$$F'(\nu) = \frac{A}{(\alpha^{-2}-\nu)^2} + \frac{B}{(c^{-2}-\nu)(\alpha^{-2}-\nu)} \qquad (2.17)$$

The first term coincides with the solution for the case $0<\alpha<c$, but the presence of the second term shows that the problem is now indeterminate. For we obtain, instead of (2.12), Equation

$$-\sigma_z^\circ = 8\mu b^2\pi\Big[A\int_0^\infty \frac{(\nu+{}^1/_2\, b^{-2})^2-\nu\sqrt{a^{-2}+\nu}\sqrt{b^{-2}+\nu}}{(\alpha^{-2}+\nu)^2\sqrt{\alpha^{-2}+\nu}}\,d\nu + {}$$
$$+ B\int_0^\infty \frac{(\nu+{}^1/_2 b^{-2})^2-\nu\sqrt{a^{-2}+\nu}\sqrt{b^{-2}+\nu}}{(c^{-2}+\nu)\,(\alpha^{-2}+\nu)\sqrt{a^{-2}+\nu}}\,d\nu\Big] \tag{2.18}$$

which is insufficient to determine the two constants A and B.

In the case $b < \alpha < a$, the function $F'(\nu)$ is constructed in a more complicated way, but it can be shown in this case that the stresses cannot have a singularity of order $\frac{1}{2}$ at the edge of the crack. For this reason we shall not dwell further on this case.

The solution obtained for the case $0 < \alpha < c$ is single valued so that the velocity of propagation of the crack, α, is determined as a function of the initial stress σ_z° from the condition (1.6). In the following section we shall see that for $c < \alpha < b$ the condition (1.6) cannot be satisfied since the integral on the right-hand side of (1.6) proves to be negative.

3. Behavior of the solution near the edge of the crack. The asymptotic behavior of the solution near the edge of the crack ($r = \alpha t$, $z = 0$) is determined by the singularity of $F'(\nu)$ at the point $\nu = 1/\alpha^2$. Since in (2.17) the second term has a weaker singularity at this point than the first term, the former makes no contribution to the leading term of the asymptotic expansion of the solution near the edge. Therefore, the asymptotic behavior of the solution must be studied with $F'(\nu)$ in the form (2.7) for $0 < \alpha < b$.

First of all we calculate the original functions (2.4). However, since only the behavior of the solution near the point $r = \alpha t$, $z = 0$ is important, it suffices to calculate the first terms of the asymptotic expansions of these functions near the points $\vartheta^{(1,2)} = \pm 1/\alpha$. To do this we proceed as follows:

$$V_\xi^{(1)}(\vartheta^{(1)}) = A\int_0^{\vartheta^{(1)}} \frac{2\vartheta^2(1-2b^2\vartheta^2)\,d\vartheta}{\sqrt{a^{-2}-\vartheta^2}\,(\alpha^{-2}-\vartheta^2)^2} = \frac{2A\vartheta^{(1)\,2}(1-2b^2\vartheta^{(1)\,2})}{\sqrt{a^{-2}-\vartheta^{(1)\,2}}}\int_0^{\vartheta^{(1)}}\frac{d\vartheta}{(\alpha^{-2}-\vartheta^2)^2} +$$
$$+ 2A\int_0^{\vartheta^{(1)}}\left[\frac{\vartheta^2(1-2b^2\vartheta^2)}{\sqrt{a^{-2}-\vartheta^2}} - \frac{\vartheta^{(1)\,2}(1-2b^2\vartheta^{(12)})}{\sqrt{a^{-2}-\vartheta^{(1)\,2}}}\right]\frac{d\vartheta}{(\alpha^{-2}-\vartheta^2)^2}$$

The second term has a singularity no greater than a logarithmic one at the points $\vartheta^{(1)} = \pm 1/\alpha$ while the first term has a pole of order one. Thus

$$V_\xi^{(1)}(\vartheta^{(1)}) \approx \frac{\alpha^2 A\vartheta^{(1)\,3}(1-2b^2\vartheta^{(1)\,2})}{(\alpha^{-2}-\vartheta^{(1)\,2})\sqrt{a^{-2}-\vartheta^{(1)\,2}}} \tag{3.1}$$

Analogously,

$$V_\xi^{(2)}(\vartheta^{(2)}) = -\frac{2\alpha^2 b^2 A\vartheta^{(2)\,2}\sqrt{b^{-2}-\vartheta^{(2)\,2}}}{\alpha^{-2}-\vartheta^{(2)\,2}} + O\left(\ln\left(1-\alpha^2\vartheta^{(2)\,2}\right)\right) \tag{3.2}$$

$$V_z^{(1)}(\vartheta^{(1)}) = \frac{\alpha^2 A \vartheta^{(1)2}(1 - 2b^2\vartheta^{(1)2})}{\alpha^{-2} - \vartheta^{(1)2}} + O(\ln(1 - \alpha^2\vartheta^{(1)2})) \qquad (3.2)\ \text{(cont)}$$

$$V_z^{(2)}(\vartheta^{(2)}) = \frac{2\alpha^2 b^2 A \vartheta^{(2)4}}{\alpha^{-2} - \vartheta^{(2)2}} + O(\ln(1 - \alpha^2\vartheta^{(2)2}))$$

$$\frac{1}{2\mu}\Sigma_z^{(1)}(\vartheta^{(1)}) = -\frac{2\alpha^2 b^2 A \vartheta^{(1)2}(\vartheta^{(1)2} - {}^1/_2\, b^{-2})^2}{\sqrt{a^{-2} - \vartheta^{(1)2}}\,(\alpha^{-2} - \vartheta^{(1)2})} + O(\ln(1 - \alpha^2\vartheta^{(1)2}))$$

$$\frac{1}{2\mu}\Sigma_z^{(2)}(\vartheta^{(2)}) = -\frac{2\alpha^2 b^2 A \vartheta^{(2)4}\sqrt{b^{-2} - \vartheta^{(2)2}}}{\alpha^{-2} - \vartheta^{(2)2}} + O(\ln(1 - \alpha^2\vartheta^{(2)2}))$$

$$\frac{1}{2\mu}T_{\xi z}^{(1)}(\vartheta^{(1)}) = -\frac{\alpha^2 A \vartheta^{(1)3}(1 - 2b^2\vartheta^{(1)2})}{\alpha^{-2} - \vartheta_p^2} + O(\ln(1 - \alpha^2\vartheta^{(1)2}))$$

$$\frac{1}{2\mu}T_{\xi z}^{(2)}(\vartheta^{(2)}) = \frac{\alpha^2 A \vartheta^{(2)3}(1 - 2b^2\vartheta^{(2)2})}{\alpha^{-2} - \vartheta^{(2)2}} + O(\ln(1 - \alpha^2\vartheta^{(2)2}))$$

We can now write

$$v_r^{(1)} \approx \alpha^2 A \int_{-\pi}^{\pi} \frac{\vartheta^{(1)3}(1 - 2b^2\vartheta^{(1)2})}{(\alpha^{-2} - \vartheta^{(1)2})\sqrt{a^{-2} - \vartheta^{(1)2}}} \cos\Omega\, d\Omega \qquad (3.3)$$

It is convenient here to transform to the variable integration $\nu = \vartheta^{(1)2}$. We then obtain

$$v_r^{(1)} \approx A \int_{l^{(1)}} \frac{(1 - 2b^2\nu)\, d\nu}{(\alpha^{-2} - \nu)\sqrt{a^{-2} - \nu}\sqrt{\nu - \alpha^{-2}(1 - 2\eta - 2\alpha\zeta\sqrt{a^{-2} - \nu}}} \qquad (3.4)$$

correct up to terms of order $\ln|z|$ and $\ln|\alpha t - r|$.

Here $\eta = (r - \alpha t)/\alpha t$, $\zeta = z/\alpha t$, and the path $l^{(1)}$ is shown in Fig.3.

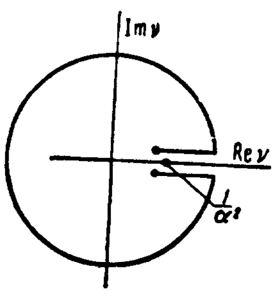

Fig. 3

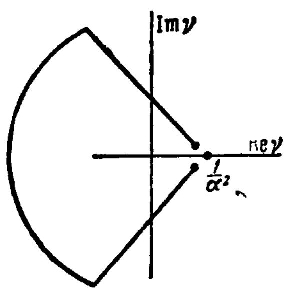

Fig. 4

The ends of the path are at the points

$$\nu_{1,2}^{(1)} = \alpha^{-2}(1 - 2\eta \pm 2i\zeta\sqrt{1 - \alpha^2 a^{-2}})$$

The principal value of the last radical in the numerator of the integrand of (3.4) must be taken so that it has a positive imaginary value for $\nu = 0$. The path $l^{(1)}$ may be deformed as shown in Fig.4. It is now clear that the asymptotic behavior of the integral for $\eta, \zeta \to 0$ is determined by the behavior of the integrand on the straight-line portions of the contour near the points $\nu_1^{(1)}$ and $\nu_2^{(2)}$, and we easily obtain

$$v_r^{(1)} \approx -\frac{\sqrt{2}\,\pi A(\alpha^2-2b^2)}{\sqrt{1-\alpha^2a^{-2}}}\operatorname{Re}\frac{1}{\sqrt{\eta-i\zeta\sqrt{1-\alpha^2a^{-2}}}} \tag{3.5}$$

Analogously,

$$v_r^{(2)} = -2\sqrt{2}\pi b^2 A\sqrt{1-\alpha^2b^{-2}}\operatorname{Re}\frac{1}{\sqrt{\eta-i\zeta\sqrt{\alpha^2b^{-2}}}}$$

$$v_z^{(1)} \approx -\sqrt{2}\pi A(\alpha^2-2b^2)\operatorname{Im}\frac{1}{\sqrt{\eta-i\zeta\sqrt{1-\alpha^2a^{-2}}}}$$

$$v_z^{(2)} \approx -2\sqrt{2}\pi b^2 A\operatorname{Im}\frac{1}{\sqrt{\eta-i\zeta\sqrt{1-\alpha^2b^{-2}}}}$$

$$\frac{1}{2\mu}\tau_{rz}^{(1)} \approx \sqrt{2}\pi\alpha^{-1}A(\alpha^2-2b^2)\operatorname{Im}\frac{1}{\sqrt{\eta-i\zeta\sqrt{1-\alpha^2a^{-2}}}} \tag{3.6}$$

$$\frac{1}{2\mu}\tau_{rz}^{(2)} \approx -\sqrt{2}\pi\alpha^{-1}A(\alpha^2-2b^2)\operatorname{Im}\frac{1}{\sqrt{\eta-i\zeta\sqrt{1-\alpha^2b^{-2}}}}$$

$$\frac{1}{2\mu}\sigma_z^{(1)} \approx \frac{\sqrt{2}\,\pi A(\alpha^2-2b^2)^2}{2b^2\alpha\sqrt{1-\alpha^2a^{-2}}}\operatorname{Re}\frac{1}{\sqrt{\eta-i\zeta\sqrt{1-\alpha^2a^{-2}}}}$$

$$\frac{1}{2\mu}\sigma_z^{(2)} \approx -2\sqrt{2}\pi\alpha^{-1}b^2A\sqrt{1-\alpha^2b^{-2}}\operatorname{Re}\frac{1}{\sqrt{\eta-i\zeta\sqrt{1-\alpha^2b^{-2}}}}$$

We now turn to condition (1.6). Let us take the contour l_δ in the form of a small rectangle with sides $2\delta_1$ and $2\delta_2$ in the r and z directions, respectively, and let δ_2 go to zero for fixed δ_1. Then the integrals on the sides parallel to the z-axis vanish and the left-hand side of (1.6) takes the form

$$2\lim_{\delta_1\to 0}\lim_{\delta_2\to 0}\int_{\alpha t-\delta_1}^{\alpha t+\delta_1}(\tau_{rz}v_r+\sigma_z v_z)\,dr$$

It is easy to see that τ_{rz} has no singularity at $r=\alpha t$, $z=0$, so that the integral of the first term may be dropped. With the aid of (3.6) we obtain

$$8\mu\pi^3b^2A^2\,\frac{\sqrt{1-\alpha^2a^{-2}}\sqrt{1-\alpha^2b^{-2}}-(1-{}^1\!/_2\alpha^2b^{-2})^2}{\sqrt{1-\alpha^2a^{-2}}}=C \tag{3.7}$$

It is clear from this that α cannot be greater than or equal to c, since the left-hand side changes sign at $\alpha=c$. Substituting the value of A from (2.12), we have

$$\frac{\pi(\sigma_z^\circ)^2\left[\sqrt{1-\alpha^2a^{-2}}\sqrt{1-\alpha^2b^{-2}}-(1-{}^1\!/_2\alpha^2b^{-2})^2\right]}{8\mu b^2[J(\alpha)]^2\sqrt{1-\alpha^2a^{-2}}}=C \tag{3.8}$$

where

$$J(\alpha)=\int_0^\infty\frac{(\nu+{}^1\!/_2b^{-2})^2-\nu\sqrt{\alpha^{-2}+\nu}\sqrt{b^{-2}+\nu}}{(\alpha^{-2}+\nu)^2\sqrt{a^{-2}+\nu}}\,d\nu \tag{3.9}$$

Equation (3.8) determines the velocity of propagation of the crack as a function of the initial stress. For $\alpha \to 0$, we have

$$J(\alpha) \approx \frac{\pi (a^2 - b^2)}{4\alpha^2 b^2} \alpha$$

and, therefore, for small α we can write

$$\frac{a^2 [\sigma_z^\circ (0)]^2}{\mu\pi (a^2 - b^2)} = C \tag{3.10}$$

This gives the smallest value $\sigma_z^\circ(0)$ for which the development of a crack can begin

$$\sigma_z^\circ (0) = a^{-1} \sqrt{\mu\pi (a^2 - b^2) C} \tag{3.11}$$

At this point the problem merges with the theory of equilibrium cracks. Strictly speaking, the problem assumes the presence of a small crack at the initial time, but the dimensions of the crack are not taken into account in the mathematical formulation. If this situation is considered, it may be concluded that $\sigma_z^\circ(0)$ coincides with the limiting stress for the initial crack. This allows us to express the constant C in terms of the static modulus of cohesion K (see, e.g., [7]). Using the solution of the static problem [8]

$$\sigma_z = \frac{2\sigma_z^\circ r_0}{\pi \sqrt{r^2 - r_0^2}} - \frac{2\sigma_z^\circ}{\pi} \sin^{-1} \frac{r_0}{r} \quad \text{for} \quad z = 0,\ r > r_0$$

where r_0 is the radius of the initial crack and using the criterion of equilibrium [7], we obtain

$$\sigma_z^\circ (0) = \frac{K}{\sqrt{2r_0}}$$

This value must be equal to that from (3.11), whence

$$C = \frac{K^2 a^2}{2\mu\pi (a^2 - b^2) r_0} \tag{3.12}$$

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EDITORIAL NOTE

*) The chapter mentioned on page 796 occurs only in the Russian edition of this book.